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Dual state vector of nonlinear coherent state and its application in complex P-representation

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Abstract
We introduce the dual vector $|z, f(n)\rangle_*$ of a single-mode nonlinear coherent state (NCS) by using the contour integral form of δ-function, and the explicit form of $|z, f(n)\rangle_*$ is obtained in enlarged Hilbert space. The nonlinear Fock states and their contour representation are deduced. A new completeness relation composed of $|z, f(n)\rangle_*$ and NCS in contour integral form is derived and its application in constructing a complex P-representation of density operators is shown.

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1. Introduction

As is well known, a coherent state is the eigenket of the Bose annihilation operator a [1]. As to the eigenkets of creation operator $a^\dagger$, for some time it had been believed that the eigenket of a creation operator was identically zero [2]. In [3] Fan *et al* pointed out that such an assertion is not rigorous. In fact, the eigenket $|z\rangle_*$ of the creation operator, as a dual vector of the normal coherent state $|z\rangle$, was derived in [3] by using Heitler's contour integral form of δ-function [4], and its properties were studied. Later, Klauder [5] noted that this formulation complies with Dirac's ξ-representation of a harmonic oscillator [6], which is termed Dirac's contour representation by Vourdas and Bishop [7]. Vourdas and Bishop employed this representation to discuss the analytic continuation of various physical quantities into the negative temperature region [8], and Wünsche also independently derived the eigenfunction of creation operator [9, 10].

Recently, the nonlinear coherent state (NCS) [11–19], which is generalization of the coherent state, has attracted much attention. Arik and Coon [11] were the first to tackle NCS; further studies by various authors can be found in [12, 13]. Single-mode NCS was defined as the eigenvector of $f(N)a$, $N = a^\dagger a$. In the linear limit, $f(N) = 1$, the NCS becomes the

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usual coherent state $|z\rangle$. Many already known states, such as the squeezed state and negative binomial state, can be regarded as some kind of NCS [14]. Recently, a class of NCS has been realized physically as the stationary states of the centre-of-mass motion of a trapped ion [15]. Note also that some of the coherent states which are eigenstates of $f(N)a$ are just particular cases of those associated with the higher-order supersymmetric partners of the harmonic oscillator [16, 17] (see also the recent discussion in [18]).

A question naturally arises, what is the dual state vector of the eigenket of $f(N)a$? In this paper, we will demonstrate that the dual state of NCS is the eigenket of $a^\dagger f(N) = f(N-1)a^\dagger$. Thus, we search for the eigenvector of $f(N)a^\dagger$, and then discuss its properties and applications. We shall make full use of the contour integration form of δ-function for realizing our goal.

2. Eigenket of $f(N)a^\dagger$

We denote the eigenvector of $f(N)a^\dagger$ as $|z, f(n)\rangle_*$, which satisfies

$$f(N)a^\dagger\,|z, f(n)\rangle_* = z^*\,|z, f(n)\rangle_* \qquad n = 0, 1, 2, \ldots \tag{1}$$

where z^* is a complex number, and the subscript $_*$ only denotes that this ket belongs to $f(N)a^\dagger$. By expanding $|z, f(n)\rangle_*$ in Fock space

$$|z, f(n)\rangle_* = \sum_{n=0}^{\infty} C_n\,|n\rangle \qquad C_n = \langle n|z, f(n)\rangle_* \tag{2}$$

where $|n\rangle$ is a number state $|n\rangle = a^{\dagger n}/\sqrt{n!}\,|0\rangle$, and substituting it into equation (1) we have

$$\sum_{n=0}^{\infty} C_n f(n+1)\sqrt{n+1}\,|n+1\rangle = z^* \sum_{m=0}^{\infty} C_n\,|n\rangle$$

from which we get the recursive relation

$$\begin{aligned} &z^*C_0 = 0 \qquad z^*C_1 = C_0 f(1) \qquad z^{*2}C_2 = C_0\sqrt{1\cdot 2}f(1)f(2) \\ &\ldots \qquad z^{*n}C_n = C_0\sqrt{n!}\prod_{i=1}^{n} f(i). \end{aligned} \tag{3}$$

By analogy with the property of Dirac's delta function $x\delta(x) = 0$, where x is real, we see that the solution of C_0 is

$$C_0 = \delta(z^*) \tag{4}$$

where $\delta(z^*)$ is a delta function of complex argument. Then a question naturally arises: how to define such kind of delta function? We appeal to Heitler's contour integral expression of δ-function [4]:

$$\delta(z^*) = \left.\frac{1}{2\pi \mathrm{i} z^*}\right|_{\mathrm{c}} \tag{5}$$

where the symbol $|_{\mathrm{c}}$ means a closed counter-clockwise contour which encircles the point $z^* = 0$. In fact, equation (5) is deduced by comparing the Cauchy integral formula

$$g(0) = \frac{1}{2\pi \mathrm{i}} \oint_{\mathrm{c}} \frac{g(z^*)}{z^*}\,\mathrm{d}z^*$$

and the usual definition of the real Dirac δ-function

$$g(0) = \int_{-\infty}^{\infty} g(x)\delta(x)\,\mathrm{d}x.$$

It is understood that the symbol $|_c$ means that the subsequent integration over z^* must be carried out along the path C. Substituting equation (3) into (2) we get the form of $|z, f(n)\rangle_*$,

$$|z, f(n)\rangle_* = \sum_{n=0}^{\infty}\left[\frac{\delta(z^*)}{(z^*)^n}\prod_{i=1}^{n} f(\mathrm{i})\right]a^{\dagger n}\,|0\rangle = \sum_{n=0}^{\infty}\left[\frac{\delta(z^*)}{(z^*)^n}a^{\dagger n}\prod_{i=1}^{n} f(N+\mathrm{i})\right]|0\rangle. \qquad (6)$$

Further, using the operator identity

$$\left(a^\dagger\right)^n f(N+1)\ldots f(N+n) = \left[f(N)a^\dagger\right]^n$$

and the higher-order derivatives of the δ-function

$$\delta^{(n)}(z^*) = \left(\frac{\mathrm{d}}{\mathrm{d}z^*}\right)^n \delta(z^*) = (-)^n n!\frac{\delta(z^*)}{(z^*)^n}$$

which is also understood in the meaning of contour integration, equation (6) becomes

$$\begin{aligned}|z, f(n)\rangle_* &= \sum_{n=0}^{\infty}\left[\frac{\delta(z^*)}{(z^*)^n}(f(N)a^\dagger)^n\right]|0\rangle = \sum_{n=0}^{\infty}\frac{1}{n!}\left(-f(N)a^\dagger\right)^n\delta^{(n)}(z^*)\,|0\rangle \\ &= \exp\left(-f(N)a^\dagger\frac{\mathrm{d}}{\mathrm{d}z^*}\right)\delta(z^*)\,|0\rangle = \delta\left[z^* - f(N)a^\dagger\right]|0\rangle\,. \end{aligned} \qquad (7)$$

In particular, when $f(N) = 1$, $|z, f(n)\rangle_*$ reduces to the eigenket $|z\rangle_*$ of $a^\dagger$ [3]

$$|z\rangle_* = \delta(z^* - a^\dagger)|0\rangle. \qquad (8)$$

It is noteworthy that $|z\rangle_*$ and the un-normalized usual coherent state $\langle z\| = \langle 0|\exp(z^*a)$ span a completeness relation in the meaning of contour integration

$$\begin{aligned}\oint_{c^*}|z\rangle_*\ \langle z\|\,\mathrm{d}z^* &= \oint_{c^*}\mathrm{d}z^*\,\delta(z^*-a^\dagger)\,|0\rangle\,\langle 0|\exp(z^*a) \\ &= \oint_{c^*}\mathrm{d}z^*\,\delta(z^*-a^\dagger) : \mathrm{e}^{-a^\dagger a} : \exp(z^*a) \\ &=: \mathrm{e}^{a^\dagger a - a^\dagger a} := 1. \end{aligned} \qquad (9)$$

Note we have used the normal product form of vacuum projector

$$|0\rangle\,\langle 0| =: \mathrm{e}^{-a^\dagger a} : .$$

In [20] this *unity of resolution* in (9) is used to construct the *complex P-representation* of density operators, which was first proposed by Gardiner [21] in quantum optics theory.

3. The completeness relation of $|z, f(n)\rangle_*$

Now we study what is the completeness relation regarding to the dual vector of NCS. Let the un-normalized NCS $\|z, f(n)\rangle$ be the eigenvector of $f(N)a$, which satisfies

$$f(N)a\,\|z, f(n)\rangle = z\,\|z, f(n)\rangle \qquad (10)$$

with its explicit form being

$$\|z, f(n)\rangle = \exp\left(\frac{z}{f(N-1)}a^\dagger\right)|0\rangle = \sum_{n=0}^{\infty}\frac{z^n}{n!}\left[\frac{1}{f(N-1)}a^\dagger\right]^n|0\rangle. \qquad (11)$$

The dual vector of $\|z, f(n+1)\rangle$ is the eigenvector of $f(N)a^\dagger$ (we have emphasized that the dual state of NCS of $f(N)a$ is the eigenket of $a^\dagger f(N) = f(N-1)a^\dagger$), because only such a correspondence can result in equations (12) and (13). In fact, with use of equations (7) and (11) we have

$$\langle z, f(n+1)\| z', f(n)\rangle_* = \langle 0| \sum_{n=0}^{\infty} \frac{z^{*n}}{n!} \left[a \frac{1}{f(N)} \right]^n \sum_{n'=0}^{\infty} \frac{1}{n'!} \big(- f(N) a^\dagger \big)^{n'} \delta^{(n')}(z^*) \,|0\rangle$$

$$= \sum_{n=0}^{\infty} \frac{(-z^*)^n \delta^{(n)}(z'^*)}{n!} = \delta(z'^* - z^*) \tag{12}$$

or

$$_*\langle z', f(n)\, \| z, f(n+1)\rangle = \delta(z' - z). \tag{13}$$

It then follows that

$$\oint_{\mathrm{c}} \|z', f(n)\rangle\, {}_*\langle z', f(n-1)\, \|z, f(n)\rangle\, \mathrm{d}z' = \|z, f(n)\rangle$$

which is valid for any well defined $\|z, f(n)\rangle$, therefore yields the completeness relation

$$\oint_{\mathrm{c}} \|z, f(n)\rangle\ {}_*\langle z, f(n-1)|\, \mathrm{d}z = 1 \tag{14}$$

or

$$\oint_{\mathrm{c}^*} |z, f(n-1)\rangle_*\ \langle z, f(n)\|\, \mathrm{d}z^* = 1. \tag{15}$$

4. Some examples

Since many well known states in quantum optics theory can be regarded as NCS, we can find their dual vectors in enlarged Hilbert space. The most obvious example is the Susskind–Glogower (SG) phase state, the eigenket of SG phase operator $(N+1)^{-1/2}\, a$, defined as

$$(N+1)^{-1/2}\, a\, |\alpha\rangle = \alpha\, |\alpha\rangle \tag{16}$$

which coincides with the definition (10) of NCS with $f(N) = (N+1)^{-1/2}$,

$$|\alpha\rangle = \exp\big(\alpha\sqrt{N} a^\dagger\big)\, |0\rangle = \sum_{n=0}^{\infty} \frac{1}{n!} \big(\alpha\sqrt{N} a^\dagger\big)^n\, |0\rangle = \sum_{n=0}^{\infty} \alpha^n\, |n\rangle \qquad |\alpha| < 1.$$

According to equation (7), the $|\alpha\rangle$ dual state vector is

$$|\alpha\rangle_* = \delta\big[\alpha^* - (N)^{-1/2}\, a^\dagger\big]|0\rangle. \tag{17}$$

From equation (13), we have

$$_*\langle\alpha|\alpha'\rangle = \langle 0|\, \delta\big[\alpha - a\, (N)^{-1/2}\big] \exp\big[\alpha'\sqrt{N} a^\dagger\big]\, |0\rangle$$
$$= \langle 0| \exp\big[-\alpha'\sqrt{N} a^\dagger\big] \delta\big[\alpha - a\, (N)^{-1/2}\big] \exp\big[\alpha'\sqrt{N} a^\dagger\big]\, |0\rangle = \delta\big(\alpha - \alpha'\big).$$

Another example is the negative binomial state (NBS), which interpolates between the number state and SG phase state. Its standard form is

$$|\beta, s\rangle = \sum_{n=0}^{\infty} \left[\binom{n+s}{n} \beta^{n+1} (1-\beta)^n \right]^{1/2} |n\rangle. \tag{18}$$

In [22] Fan and Jing converted NBS to the form of $SU(1,1)$ coherent state by using the following Bose operator realization of $SU(1,1)$ generators:

$$R_- = a\sqrt{2\lambda - 1 + N} \qquad R_+ = \sqrt{2\lambda - 1 + N} a^\dagger \qquad R_3 = N + \lambda \tag{19}$$

which satisfy

$$[R_-, R_+] = 2R_3 \qquad [R_3, R_+] = R_+ \qquad [R_3, R_-] = -R_-. \tag{20}$$

In fact, using the relation

$$\begin{aligned}\binom{2\lambda - 1 + n}{n}^{1/2} n!\,|n\rangle &= [2\lambda\,(2\lambda + 1)\ldots(2\lambda + n - 1)]^{1/2}\sqrt{n!}\,|n\rangle \\ &= \left(\sqrt{2\lambda - 1 + N}a^\dagger\right)^n |0\rangle = R_+^n\,|0\rangle\end{aligned}$$

we see the NBS (letting $\beta = (\operatorname{sech} r)^2$, $s = 2\lambda - 1$) can be recast into

$$\begin{aligned}\sum_{n=0}^{\infty}\left(\tanh^2 r\right)^{\frac{n}{2}}\left(\operatorname{sech}^2 r\right)^{\lambda}\binom{2\lambda - 1 + n}{n}^{1/2}|n\rangle &= (\operatorname{sech} r)^{2\lambda}\sum_{n=0}^{\infty}\frac{R_+^n}{n!}(\tanh r)^n\,|n\rangle \\ &= (\operatorname{sech} r)^{2\lambda}\exp\left(R_+\tanh r\right)|0\rangle \equiv |r, \lambda\rangle.\end{aligned} \tag{21}$$

With this result in mind it is easy to prove that $|r, \lambda\rangle$ can be expressed as a NCS. With $R_3 - \lambda$ acting on $|r, \lambda\rangle$

$$(R_3 - \lambda)\,|r, \lambda\rangle = (\operatorname{sech} r)^{2\lambda}\,[R_3 - \lambda, \exp(R_+\tanh r)]\,|0\rangle = (R_+\tanh r)\,|r, \lambda\rangle$$

which is

$$(R_3 - R_+\tanh r)\,|r, \lambda\rangle = \lambda\,|r, \lambda\rangle$$

or

$$a\left(N - \sqrt{2\lambda - 1 + N}a^\dagger\tanh r\right)|r, \lambda\rangle = \left[(N + 1)a - \sqrt{2\lambda + N}(N + 1)\tanh r\right]|r, \lambda\rangle = 0$$

thereby $|r, \lambda\rangle$ is the NCS of $1/\sqrt{2\lambda + N}$,

$$\frac{1}{\sqrt{2\lambda + N}}a\,|r, \lambda\rangle = \tanh r|r, \lambda\rangle. \tag{22}$$

This result can also be obtained via another approach. Observing

$$R_- g(R_3) = g(R_3 + 1)R_- \qquad R_+ g(R_3) = g(R_3 - 1)R_+$$

and

$$\begin{aligned}R_- R_+ &= a\,(2\lambda - 1 + N)\,a^\dagger = (2\lambda + N)\,(N + 1) = (R_3 + \lambda)\,(R_3 - \lambda + 1) \\ R_+ R_- &= (R_3 - \lambda)\,(R_3 + \lambda - 1)\end{aligned}$$

there appears

$$\left[\frac{1}{R_3 + \lambda}R_-, R_+\right] = \frac{1}{R_3 + \lambda}R_- R_+ - \frac{1}{R_3 + \lambda - 1}R_+ R_- = 1. \tag{23}$$

From equations (21) and (23), we can directly write the $|r, \lambda\rangle$ as NCS

$$\frac{1}{R_3 + \lambda}R_-\,|r, \lambda\rangle = \tanh r\,|r, \lambda\rangle = \frac{1}{\sqrt{N + 2\lambda}}a\,|r, \lambda\rangle\,.$$

Thus, in this case, according to the definition of NCS, $f(N) = \frac{1}{\sqrt{N+2\lambda}}$. Therefore, its dual vector is

$$|r\,(z = \tanh r)\,, \lambda\rangle_* = \delta\left[z^* - \frac{1}{\sqrt{N + 2\lambda - 1}}a^\dagger\right]|0\rangle. \tag{24}$$

5. Applications of $|z, f(n)\rangle_*$ in finding nonlinear Fock states

NCS and its dual vector can be used to define contour representation of nonlinear Fock states. Note that contour representation of the usual Fock state can be expressed as

$$|n\rangle = \frac{\sqrt{n!}}{2\pi \mathrm{i}} \oint_c \frac{\mathrm{d}z}{z^{n+1}} \, \|z\rangle \tag{25}$$

and

$$\langle n| = \oint_c \mathrm{d}z \, \frac{z^n}{\sqrt{n!}} \, {}_*\langle z| \tag{26}$$

where $|n\rangle$ is the Fock state while $\|z\rangle$ is un-normalized coherent state $\|z\rangle = \mathrm{e}^{za^\dagger}|0\rangle$, and ${}_*\langle z|$ is the dual vector of $\|z\rangle$ [3]. We can define nonlinear Fock states as

$$\begin{aligned} |n, f(n)\rangle &\equiv \frac{\sqrt{n!}}{2\pi \mathrm{i}} \oint_c \frac{\mathrm{d}z}{z^{n+1}} \, \|z, f(n)\rangle \\ &= \frac{\sqrt{n!}}{2\pi \mathrm{i}} \oint_c \frac{\mathrm{d}z}{z^{n+1}} \sum_{m=0}^{\infty} \frac{z^m}{\sqrt{m!}} \left[\frac{1}{f(N-1)} a^\dagger \right]^m |0\rangle \\ &= \frac{1}{\sqrt{n!}} \left[\frac{1}{f(N-1)} a^\dagger \right]^n |0\rangle \end{aligned} \tag{27}$$

and

$$\begin{aligned} \langle\langle n, f(n)| &\equiv \oint_c \mathrm{d}z \, \frac{z^n}{\sqrt{n!}} \, {}_*\langle z, f(n-1)| \\ &= \oint_c \mathrm{d}z \, \frac{z^n}{\sqrt{n!}} \, \langle 0| \sum_{m=0}^{\infty} \frac{1}{2\pi \mathrm{i} z^{m+1}} [af(N-1)]^m \\ &= \frac{1}{\sqrt{n!}} \langle 0| \, [f(N)a]^n . \end{aligned} \tag{28}$$

It is noteworthy that $f(N)a$ and $(f(N-1))^{-1}$ satisfy

$$\left[f(N)a, \frac{1}{f(N-1)} a^\dagger \right] = 1$$

and $|n, f(n)\rangle$ and $\langle\langle n, f(n)|$ are not mutually Hermite conjugate. Nevertheless, they satisfy the completeness relation

$$\begin{aligned} \sum_{n=0}^{\infty} |n, f(n)\rangle \, \langle\langle n, f(n)| &= \sum_{n=0}^{\infty} \frac{1}{n!} \left[\frac{1}{f(n-1) \ldots f(0)} \sqrt{n!} \, |n\rangle \right] \left[\langle n| \, f(0) \ldots f(n-1) \sqrt{n!} \right] \\ &= \sum_{n=0}^{\infty} |n\rangle \, \langle n| = 1. \end{aligned} \tag{29}$$

The contour representation of $|n, f(n)\rangle$ and $\langle\langle n, f(n)|$ are nonlinear generalization of Dirac's contour representation (25) and (26) (also see [8]). In particular, when $f(N) = 1$, we get the usual contour representation of Fock state.

6. Complex P-representation of density operator in terms of NCS

We can also use the eigenket of $a^\dagger f(N)$ to construct *complex P-representation* [21] of density operator ρ. In fact, by using equations (14), (15), we have

$$\rho = \oint_c \oint_{c'^*} \mathrm{d}z \, \mathrm{d}z'^* \, \|z, f(n)\rangle \; {}_*\langle z, f(n-1)| \, \rho |z', f(n-1)\rangle_* \, \langle z', f(n)\|$$

$$= \oint_{c} \oint_{c'^*} dz\, dz'^* \frac{\| z, f(n)\rangle \langle z', f(n)\|}{\langle z', f(n)\| z, f(n)\rangle} P(z, z'^*) \tag{30}$$

where

$$\langle z', f(n)\| z, f(n)\rangle = \sum_{k=0}^{\infty} \frac{(z'^* z)^k}{k!} \left[\prod_{m=0}^{n-1} f^{-1}(m) \right]^2$$

and

$$P(z, z'^*) = \langle z', f(n)\| z, f(n)\rangle \left({}_*\langle z, f(n-1)|\, \rho \,|z', f(n-1)\rangle_* \right) \tag{31}$$

is the *complex P-representation* of density operator ρ. For example, the *complex P-representation* of the normalized NCS $|\alpha, f(n)\rangle = (\langle \alpha, f(n)\| \alpha, f(n)\rangle)^{-1/2} \|\alpha, f(n)\rangle$ is

$$\begin{aligned} P_{|\alpha, f\rangle\langle \alpha, f|}(z, z'^*) &= \langle z', f(n)\| z, f(n)\rangle ({}_*\langle z, f(n-1)|\alpha, f(n)\rangle \langle \alpha, f(n)|z', f(n-1)\rangle_*) \\ &= \delta(z-\alpha)\delta(z'^* - \alpha^*). \end{aligned} \tag{32}$$

The *complex P-representation* of pure Fock state $\rho = |n\rangle\langle n|$ can be written as

$$\begin{aligned} P_{|n\rangle\langle n|}(z, z'^*) &= \langle z', f(n)\| z, f(n)\rangle ({}_*\langle z, f(n-1)|n\rangle\langle n|z', f(n-1)\rangle_*) \\ &= \langle z', f(n)\| z, f(n)\rangle \left[\langle 0| \sum_{k=0}^{\infty} \frac{\delta(z)}{z^k} [f(N)a]^k |n\rangle\langle n| \sum_{l=0}^{\infty} \frac{\delta(z'^*)}{(z'^*)^l} [a^\dagger f(N)]^l |0\rangle \right] \\ &= \langle z', f(n)\| z, f(n)\rangle \frac{\delta(z)\delta(z'^*)}{(zz'^*)^n} \left[\prod_{m=0}^{n-1} f(m) \right]^2. \end{aligned} \tag{33}$$

In summary, we have constructed the eigenkets of $f(N)a^\dagger$ and examined its completeness relation. As a by-product, the nonlinear Fock states are also derived. These eigenkets may have potential uses even though they lie outside the Hilbert space (recall that the often-used coordinate (or momentum) eigenstates also lie outside Hilbert space). In like manner we feel that Dirac's work on the ξ-representation of a harmonic oscillator should receive more attention.

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